Variational perturbation theory for Markov processes

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We develop a convergent variational perturbation theory for conditional probability densities of Markov processes. The power of the theory is illustrated by applying it to the diffusion of a particle in an anharmonic potential.

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I. INTRODUCTION

Variational perturbation theory [1] transforms divergent perturbation expansions into convergent ones, with the convergence extending to infinitely strong couplings. The theory has been developed for the path integral representation of the free energy and the density matrix in quantum statistics, and has been tested for many systems, in particular, the anharmonic oscillator and the hydrogen atom, with and without a homogeneous magnetic field [1–7]. The procedure is based on approximating a potential by a local trial oscillator whose frequency is optimized order by order for each set of external end points. Recently, variational perturbation theory has also been successfully extended to statistical field theory to calculate highly accurate critical properties of second order phase transitions [8].

In this paper we develop a similarly convergent variational perturbation theory for the path-integral representation of the conditional probability density of Markov processes. In close analogy with the previous method, we approximate a given stochastic process by a trial Brownian motion with a linear drift coefficient, and optimize the damping constant. We illustrate the procedure by calculating the time dependence of the conditional probability density for a nonlinear stochastic model. After some introductory remarks on Markov processes in Sec. II, the path integral for the conditional probability density is treated perturbatively in Sec. III, and evaluated variationally in Sec. IV.

II. MARKOV PROCESSES

We start by summarizing the basic properties of Markov processes [9–12] needed in the sequel.

A. Fokker-Planck equation

A Markov process for a single stochastic variable x is characterized by the conditional probability density $P(x_b t_b|x_a t_a)$ that the event x_b is realized at the time t_b once the event x_a has taken place at time t_a. It has the initial condition

$$P(x_b t_a|x_a t_a)=\delta(x_b-x_a) \tag{1}$$

and obeys, for additive noise, the Fokker-Planck equation

$$\frac{\partial}{\partial t_b}P(x_b t_b|x_a t_a)=\hat{L}_{\rm FP}(x_b)P(x_b t_b|x_a t_a) \tag{2}$$

with the Fokker-Planck operator

$$\hat{L}_{\rm FP}(x_b)\cdot=-\frac{\partial}{\partial x_b}[K(x_b)\cdot]+D\frac{\partial^2}{\partial x_b^2}\cdot, \tag{3}$$

where $K(x)$ and D denote the drift coefficient and the diffusion coefficient, respectively. An important example is provided by the overdamped motion of a Brownian particle with mass M and friction constant κ in an external potential $V(x)$. In this case, the drift coefficient reads

$$K(x)=-\frac{V'(x)}{M\kappa}, \tag{4}$$

and the diffusion coefficient is proportional to the temperature T via Einstein's relation

$$D=M\kappa k_B T. \tag{5}$$

Since the spatial derivatives in the Fokker-Planck equation (2), (3) are all on the left-hand side, they guarantee the probability conservation

$$\frac{\partial}{\partial t_b}\int_{-\infty}^{+\infty}dx_b P(x_b t_b|x_a t_a)=0, \tag{6}$$

such that the normalization integral, which is unity at the initial time $t_b=t_a$ due to Eq. (1), remains so for all times:

$$\int_{-\infty}^{+\infty}dx_b P(x_b t_b|x_a t_a)=1. \tag{7}$$

In the long-time limit $t_b\to\infty$, the conditional probability density $P(x_b t_b|x_a t_a)$ becomes *stationary:*

$$\lim_{t_b\to\infty}P(x_b t_b|x_a t_a)=P_{\rm st}(x_b), \tag{8}$$

where $P_{\rm st}(x_b)$ denotes the time-independent solution of the Fokker-Planck equation (2), defined by

$$\hat{L}_{\rm FP}(x_b)P_{\rm st}(x_b)=0. \tag{9}$$

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By applying the Fokker-Planck operator (3), we verify that the solution is

$$P_{\rm st}(x_b)=N\exp\left[\frac{1}{D}\int^{x_b}dx\ K(x)\right], \tag{10}$$

where the normalization constant N follows from Eqs. (7) and (8):

$$N=\left\{\int_{-\infty}^{+\infty}dx_b\exp\left[\frac{1}{D}\int^{x_b}dx\ K(x)\right]\right\}^{-1}. \tag{11}$$

B. Path integral

The solution of the Fokker-Planck initial value problem (1)–(3) has the path-integral representation

$$P(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\ e^{-\mathcal{A}[x]} \tag{12}$$

with the generalized Onsager-Machlup functional

$$\mathcal{A}[x]=\frac{1}{4D}\int_{t_a}^{t_b}dt\{\dot{x}(t)-K(x(t))\}^2+\frac{1}{2}\int_{t_a}^{t_b}dtK'(x(t)), \tag{13}$$

where all paths $x(t)$ contribute in connecting the spacetime points (x_a,t_a) and (x_b,t_b). The extra term in Eq. (13) is needed since the path integral (12),(13) is by definition symmetrically ordered in the product of $\dot{x}(t)$ and $K(x(t))$, corresponding to a midpoint discretization [1]:

$$\begin{aligned}P(x_bt_b|x_at_a)=\lim_{\epsilon\to 0}&\left\{\prod_{n=1}^{N}\int_{-\infty}^{+\infty}dx_n\right\}\left(\frac{1}{4\pi D\epsilon}\right)^{N+1/2}\\&\times\exp\left\{-\frac{\epsilon}{4D}\sum_{n=1}^{N+1}\left[\frac{x_n-x_{n-1}}{\epsilon}\right.\right.\\&\left.\left.-K\left(\frac{x_n+x_{n-1}}{2}\right)\right]^2-\frac{\epsilon}{2}\sum_{n=1}^{N+1}K'\left(\frac{x_n+x_{n-1}}{2}\right)\right\}.\end{aligned} \tag{14}$$

Making use of the stationary solution (10), the path integral (12),(13) can be factorized as

$$P(x_bt_b|x_at_a)=\sqrt{\frac{P_{\rm st}(x_b)}{P_{\rm st}(x_a)}}\,\widetilde{P}(x_bt_b|x_at_a) \tag{15}$$

with the remaining path integral

$$\begin{aligned}\widetilde{P}(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\exp\left(-\frac{1}{2D}\int_{t_a}^{t_b}dt\{\tfrac{1}{2}\dot{x}^2(t)\right.\\\left.+\tfrac{1}{2}K^2[x(t)]+DK'[x(t)]\}\right).\end{aligned} \tag{16}$$

This coincides with the quantum-statistical, imaginary-time evolution amplitude [1]

$$\begin{aligned}(x_b\hbar\beta|x_a0)=\int_{x(0)=x_a}^{x(\hbar\beta)=x_b}\mathcal{D}x\exp\left(-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\left\{\frac{M}{2}\dot{x}^2(\tau)\right.\right.\\\left.\left.+U[x(\tau)]\right\}\right),\end{aligned} \tag{17}$$

in which we identify

$$t_a\equiv 0,\quad t_b\equiv\hbar\beta,\quad D\equiv\frac{\hbar}{2M} \tag{18}$$

and set

$$U(x)\equiv\frac{M}{2}K^2(x)+\frac{\hbar}{2}K'(x). \tag{19}$$

To the path integral (17), we can directly apply the known variational perturbation theory [1], which, in the present context, will lead us to a solution of the Fokker-Planck initial value problem (1)–(3).

C. Brownian motion

A solvable trial path integral is provided by the Brownian motion with a linear drift coefficient

$$K(x)=-\kappa x, \tag{20}$$

where the stationary solution (10),(11) reads

$$P_{\kappa,{\rm st}}(x)=\sqrt{\frac{\kappa}{2\pi D}}\exp\left(-\frac{\kappa}{2D}x^2\right). \tag{21}$$

Therefore, according to Eq. (15), the conditional probability density factorizes as

$$\begin{aligned}P_\kappa(x_bt_b|x_at_a)=&\exp\left[\frac{\kappa}{2}(t_b-t_a)-\frac{\kappa}{4D}(x_b^2-x_a^2)\right]\\&\times\widetilde{P}_\kappa(x_bt_b|x_at_a),\end{aligned} \tag{22}$$

and the remaining path integral is simply

$$\begin{aligned}\widetilde{P}_\kappa(x_bt_b|x_at_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\exp\left(-\frac{1}{2D}\int_{t_a}^{t_b}dt[\tfrac{1}{2}\dot{x}^2(t)\right.\\\left.+\tfrac{1}{2}\kappa^2x^2(t)]\right).\end{aligned} \tag{23}$$

This describes a quantum-statistical harmonic oscillator with the potential $U(x)=M\kappa^2x^2/2$. Inserting the imaginary-time evolution amplitude of the harmonic oscillator [13], and taking into account the identification (18), we obtain

$$\widetilde{P}_\kappa(x_b t_b|x_a t_a)=\sqrt{\frac{\kappa}{4\pi D\sinh\kappa(t_b-t_a)}}\times\exp\left\{\frac{-\kappa}{4D\sinh\kappa(t_b-t_a)}\times[(x_a^2+x_b^2)\cosh\kappa(t_b-t_a)-2x_a x_b]\right\}. \quad (24)$$

The resulting conditional probability density of the Brownian motion (20) follows from Eqs. (22),(24) and leads to the well known Gaussian distribution

$$P_\kappa(x_b t_b|x_a t_a)=\frac{1}{\sqrt{2\pi\sigma^2(x_a,t_a;t_b)}}\times\exp\left\{-\frac{[x_b-\overline{x}(x_a,t_a;t_b)]^2}{2\sigma^2(x_a,t_a;t_b)}\right\} \quad (25)$$

with the average point

$$\overline{x}(x_a,t_a;t_b)=x_a e^{-\kappa(t_b-t_a)}, \quad (26)$$

and the width

$$\sigma(x_a,t_a;t_b)=\sqrt{\frac{D}{\kappa}[1-e^{-2\kappa(t_b-t_a)}]}. \quad (27)$$

It can easily be verified that the conditional probability density (25)–(27) satisfies the initial condition (1), and obeys the Fokker-Planck equation associated with the drift coefficient (20):

$$\frac{\partial}{\partial t_b}P_\kappa(x_b t_b|x_a t_a)=\frac{\partial}{\partial x_b}[\kappa x_b P_\kappa(x_b t_b|x_a t_a)]+D\frac{\partial^2}{\partial x_b^2}P_\kappa(x_b t_b|x_a t_a). \quad (28)$$

In general, the drift coefficient $K(x)$ of a Markov process is a nonlinear function in x, and the corresponding conditional probability density $P(x_b t_b|x_a t_a)$ cannot be calculated exactly. We must then resort to approximation methods. In this paper we want to show that variational perturbation theory is a very efficient one. For this we first need an ordinary perturbation expansion, which will be derived in the next section for a typical example.

III. PERTURBATION THEORY

Consider the nonlinear drift coefficient

$$K(x)=-\kappa x-gx^3 \quad (29)$$

with a coupling constant g. Such a stochastic model is useful, for instance, to describe the statistical properties of laser light near the threshold in semiclassical laser theory [12,14]. In this case, the stochastic variable x is identified with the electric field. The parameter κ is proportional to the difference between the pump parameter σ and its threshold value $\sigma_{\rm thr}$. The coupling constant g describes the interaction between light and matter within the dipole approximation, and the diffusion constant D characterizes the spontaneous emission of radiation. For such a stochastic process, the stationary solution (10),(11) reads

$$P_{\rm st}(x)=\sqrt{\frac{2g}{\kappa}}\frac{\exp\left[-\frac{1}{2D}\left(\frac{\kappa^2}{4g}+\kappa x^2+\frac{g}{2}x^4\right)\right]}{K_{1/4}\left(\frac{\kappa^2}{8Dg}\right)}, \quad (30)$$

where $K_\nu(z)$ denotes a modified Bessel function [15]. The path integral for the conditional probability density corresponding to (12),(13) reads

$$P(x_b t_b|x_a t_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\exp\left(-\frac{1}{4D}\int_{t_a}^{t_b}dt[\dot{x}(t)+\kappa x(t)+gx^3(t)]^2+\frac{1}{2}\int_{t_a}^{t_b}dt[\kappa+3gx^2(t)]\right). \quad (31)$$

The decomposition of type (15) leads to

$$P(x_b t_b|x_a t_a)=\exp\left(\frac{\kappa}{2}(t_b-t_a)-\frac{\kappa}{4D}(x_b^2-x_a^2)-\frac{g}{8D}(x_b^4-x_a^4)\right)\widetilde{P}(x_b t_b|x_a t_a) \quad (32)$$

with the remaining path integral

$$\widetilde{P}(x_b t_b|x_a t_a)=\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\exp\left(-\frac{1}{2D}\int_{t_a}^{t_b}dt[\tfrac{1}{2}\dot{x}^2(t)+\tfrac{1}{2}\kappa^2x^2(t)+g\kappa x^4(t)+\tfrac{1}{2}g^2x^6(t)-3gDx^2(t)]\right). \quad (33)$$

For zero coupling constant g, we obtain the conditional probability density (23) of the Brownian motion (20). Expanding the exponential in powers of g, we find the first-order approximation

$$\widetilde{P}(x_b t_b|x_a t_a)=\widetilde{P}_\kappa(x_b t_b|x_a t_a)\left\{1+g\left[\frac{3}{2}\int_{t_a}^{t_b}dt\langle x^2(t)\rangle_\kappa-\frac{\kappa}{2D}\int_{t_a}^{t_b}dt\langle x^4(t)\rangle_\kappa\right]+\cdots\right\}. \quad (34)$$

On the right-hand side, we have denoted the harmonic expectation value of any functional $F[x]$ of the path $x(t)$ by

$$\langle F[x]\rangle_\kappa = \frac{1}{\widetilde{P}_\kappa(x_b t_b|x_a t_a)} \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x F[x] \times \exp\left\{ -\frac{1}{2D}\int_{t_a}^{t_b} dt \left[\frac{1}{2}\dot{x}^2(t) + \frac{1}{2}\kappa^2 x^2(t) \right] \right\}. \tag{35}$$

The latter is evaluated with the help of the generating functional for the quantum-statistical harmonic oscillator

$$\widetilde{P}_\kappa(x_b t_b|x_a t_a)[j] = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \exp\left(-\frac{1}{2D}\int_{t_a}^{t_b} dt [\tfrac{1}{2}\dot{x}^2(t) + \tfrac{1}{2}\kappa^2 x^2(t) - j(t)x(t)] \right), \tag{36}$$

which has the explicit form [13]

$$\widetilde{P}_\kappa(x_b t_b|x_a t_a)[j] = \widetilde{P}_\kappa(x_b t_b|x_a t_a) \exp\left(\frac{1}{2D}\int_{t_a}^{t_b} dt_1 x_{\rm cl}(t_1) j(t_1) + \frac{1}{4D^2}\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 G(t_1,t_2) j(t_1) j(t_2) \right). \tag{37}$$

The quantity $\widetilde{P}_\kappa(x_b t_b|x_a t_a)$ is the same as in Eq. (24), and $x_{\rm cl}(t_1)$ denotes the classical path

$$x_{\rm cl}(t_1) = \frac{x_a \sinh \kappa(t_b - t_1) + x_b \sinh \kappa(t_1 - t_a)}{\sinh \kappa(t_b - t_a)}. \tag{38}$$

The quadratic term in Eq. (37) contains the Green function

$$G(t_1,t_2) = \frac{D[\Theta(t_1-t_2)\sinh\kappa(t_b-t_1)\sinh\kappa(t_2-t_a) + \Theta(t_2-t_1)\sinh\kappa(t_b-t_2)\sinh\kappa(t_1-t_a)]}{\kappa \sinh\kappa(t_b-t_a)}. \tag{39}$$

We evaluate harmonic expectation values of polynomials in x arising from the generating functional (37) according to a slight generalization of the standard Wick theorem [16,17]. The generalization is required by the presence of the linear term in Eq. (37). The evaluation is most economically done in a recursive procedure that we illustrate with the harmonic expectation value

$$\langle x^n(\tau_1) x^m(\tau_2)\rangle_\kappa. \tag{40}$$

The procedure is as follows.

(i) Contracting $x(\tau_1)$ with $x^{n-1}(\tau_1)$ and $x^m(\tau_2)$ leads to a Green function $G(\tau_1,\tau_1)$ and $G(\tau_1,\tau_2)$ with multiplicity $n-1$ and m, respectively. The rest of the factors remains inside the expectation symbol, leading to $\langle x^{n-2}(\tau_1)x^m(\tau_2)\rangle_\kappa$ and $\langle x^{n-1}(\tau_1)x^{m-1}(\tau_2)\rangle_\kappa$.

(ii) If $n>1$, we extract one $x(\tau_1)$ from the expectation value, giving $x_{\rm cl}(\tau_1)$ multiplied by $\langle x^{n-1}(\tau_1)x^m(\tau_2)\rangle_\kappa$.

(iii) Add the terms (i) and (ii).

(iv) Repeat the previous steps until only products of expectation values $\langle x(\tau_1)\rangle_\kappa = x_{\rm cl}(\tau_1)$ remain.

With the help of this recursive procedure, the first-order harmonic expectation value $\langle x^4(\tau_1)\rangle_\kappa$ is reduced to

$$\langle x^4(\tau_1)\rangle_\kappa = x_{\rm cl}(\tau_1)\langle x^3(\tau_1)\rangle_\kappa + 3G(\tau_1,\tau_1)\langle x^2(\tau_1)\rangle_\kappa. \tag{41}$$

Similarly, we find

$$\langle x^3(\tau_1)\rangle_\kappa = x_{\rm cl}(\tau_1)\langle x^2(\tau_1)\rangle_\kappa + 2G(\tau_1,\tau_1)x_{\rm cl}(\tau_1), \tag{42}$$

and

$$\langle x^2(\tau_1)\rangle_\kappa = x_{\rm cl}^2(\tau_1) + G(\tau_1,\tau_1). \tag{43}$$

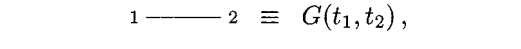
Combining Eqs. (41)–(43), we obtain in first order

$$\langle x^4(\tau_1)\rangle_\kappa = x_{\rm cl}^4(\tau_1) + 6x_{\rm cl}^2(\tau_1)G(\tau_1,\tau_1) + 3G^2(\tau_1,\tau_1). \tag{44}$$

The contractions can be represented graphically by Feynman diagrams with the following rules. Vertices represent the integrations over t:

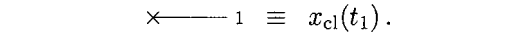
(45)

a line denotes the Green function:



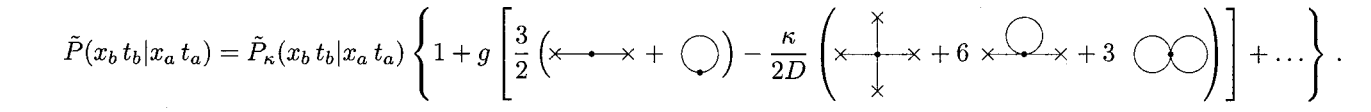
(46)

and a line ending with a cross represents the classical path:



(47)

Inserting the harmonic expectation values (43) and (44) into the perturbation expansion (34) leads to the first-order diagrams



(48)

Evaluating the diagrams, the conditional probability density for the drift coefficient (29) following from Eqs. (32) and (48) becomes, to first order in the coupling constant g,

$$P(x_b t_b|x_a t_a)=P_\kappa(x_b t_b|x_a t_a)\{1+g[c_{00}^{(1)}(\tau)+c_{20}^{(1)}(\tau)x_a^2 + c_{21}^{(1)}(\tau)x_a x_b+c_{22}^{(1)}(\tau)x_b^2+c_{40}^{(1)}(\tau)x_a^4 + c_{41}^{(1)}(\tau)x_a^3 x_b+c_{42}^{(1)}(\tau)x_a^2 x_b^2+c_{43}^{(1)}(\tau)x_a x_b^3 + c_{44}^{(1)}(\tau)x_b^4]+\cdots\}, \tag{49}$$

where the expansion coefficients $c_{ij}^{(1)}(\tau)$ are the following functions of the dimensionless variable $\tau=\kappa(t_b-t_a)$ (see Fig. 1):

$$c_{00}^{(1)}(\tau)=\frac{3D[1+4e^{-2\tau}(1-2\tau)-e^{-4\tau}(5+4\tau)]}{4\kappa^2(1-e^{-2\tau})^2},$$

$$c_{20}^{(1)}(\tau)=\frac{3[e^{-2\tau}(4\tau-5)+4e^{-4\tau}(1+2\tau)+e^{-6\tau}]}{2\kappa(1-e^{-2\tau})^3} =c_{22}^{(1)}(\tau),$$

$$c_{21}^{(1)}=\frac{3[e^{-\tau}(2-\tau)+2e^{-3\tau}(1-4\tau)-e^{-5\tau}(3\tau+4)]}{\kappa(1-e^{-2\tau})^3},$$

$$c_{40}^{(1)}(\tau)=\frac{2e^{-2\tau}+3e^{-4\tau}(1-4\tau)-6e^{-6\tau}+e^{-8\tau}}{4D(1-e^{-2\tau})^4},$$

$$c_{41}^{(1)}(\tau)=\frac{-e^{-\tau}+3e^{-3\tau}(4\tau-3)+3e^{-5\tau}(3+4\tau)+e^{-7\tau}}{2D(1-e^{-2\tau})^4} =c_{43}^{(1)}(\tau),$$

$$c_{42}^{(1)}(\tau)=\frac{3[e^{-2\tau}(3-2\tau)-8\tau e^{-4\tau}-e^{-6\tau}(3+2\tau)]}{2D(1-e^{-2\tau})^4},$$

$$c_{44}^{(1)}(\tau)=\frac{-1+6e^{-2\tau}-3e^{-4\tau}(1+4\tau)-2e^{-6\tau}}{4D(1-e^{-2\tau})^4}. \tag{50}$$

It is easy to verify that the conditional probability density (49),(50)

(i) obeys the corresponding Fokker-Planck equation following from Eqs. (2), (3), and (29):

$$\frac{\partial}{\partial t_b}P(x_b t_b|x_a t_a)=\frac{\partial}{\partial x_b}[(\kappa x_b+g x_b^3)P(x_b t_b|x_a t_a)] + D\frac{\partial^2}{\partial x_b^2}P(x_b t_b|x_a t_a), \tag{51}$$

with the initial condition (1), because of

$$c_{00}^{(1)}=c_{20}^{(1)}(0)=c_{21}^{(1)}(0)=c_{22}^{(1)}(0)=c_{41}^{(1)}(0)=c_{42}^{(1)}(0)=c_{43}^{(1)}(0) =0,$$

$$c_{40}^{(1)}(0)=-c_{44}^{(1)}(0)=\frac{1}{8D}; \tag{52}$$

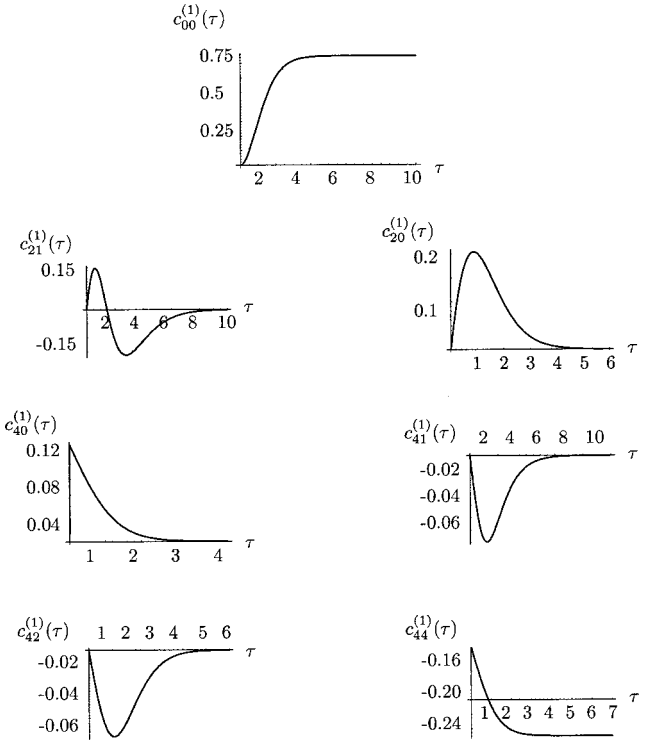


FIG. 1. Temporal behavior of the expansion coefficients (50) of the conditional probability density (49) as a function of $\tau=\kappa(t_b-t_a)$ for $D=\kappa=1$.

(ii) is normalized according to Eq. (7) for all times t_b; and

(iii) approaches the stationary solution (30) in the long-time limit (8).

All of these properties of $P(x_b t_b|x_a t_a)$ are satisfied to first order in the coupling constant g. As such, the perturbative result (49),(50) is only applicable for small values of the coupling constant g. This limitation is eliminated by a variational evaluation that enables us to find an approximate conditional probability density for all values of the coupling constant g.

IV. VARIATIONAL PERTURBATION THEORY

In quantum statistics, variational perturbation theory approximates a given potential by adding and subtracting a local trial harmonic oscillator whose frequency is then systematically optimized [1]. Here we transfer this procedure to Markov processes and modify the nonlinear drift coefficient (29) by adding and subtracting a trial Brownian motion with an as-yet unknown damping constant K:

$$K(x)=-Kx-g\left(\frac{\kappa-K}{g}x+x^3\right). \tag{53}$$

After this, we treat the combined second term as being small, of the order of the coupling constant g. The result is obtained

most easily by replacing the damping constant κ in the original drift coefficient (29), according to the substitution rule (compare with Chap. 5 in Ref. [1])

$$\kappa \to K(1+gr), \tag{54}$$

where we have introduced the abbreviation

$$r = \frac{\kappa - K}{gK}. \tag{55}$$

Writing the conditional probability density (49) as

$$P(x_b t_b | x_a t_a) = \exp[W(x_b t_b ; x_a t_a)], \tag{56}$$

we apply the substitution rule (54) to the cumulant expansion $W(x_b t_b ; x_a t_a)$, and re-expand up to the first order in the coupling constant g. Afterwards, the abbreviation r is re-expressed in terms of κ and K via Eq. (55). This leads to

$$\begin{aligned} W^{(1)}(x_b t_b ; x_a t_a ; K) = \{ & c_{00}^{(0)}(\tau) + c_{20}^{(0)}(\tau) x_a^2 + c_{21}^{(0)}(\tau) x_a x_b \\ & + c_{22}^{(0)}(\tau) x_b^2 + g[c_{00}^{(1)}(\tau) + c_{20}^{(1)}(\tau) x_a^2 \\ & + c_{21}^{(1)}(\tau) x_a x_b + c_{22}^{(1)}(\tau) x_b^2 + c_{40}^{(1)}(\tau) x_a^4 \\ & + c_{41}^{(1)}(\tau) x_a^3 x_b + c_{42}^{(1)}(\tau) x_a^2 x_b^2 \\ & + c_{43}^{(1)}(\tau) x_a x_b^3 + c_{44}^{(1)}(\tau) x_b^4] \}, \end{aligned} \tag{57}$$

where the zeroth-order expansion coefficients

$$\begin{aligned} c_{00}^{(0)}(\tau) &= \frac{1}{2} \ln \frac{K}{2\pi D(1-e^{-2\tau})} + \left(\frac{\kappa}{K} - 1 \right) \left(\frac{1}{2} - \frac{\tau e^{-2\tau}}{1-e^{-2\tau}} \right), \\ c_{20}^{(0)}(\tau) &= - \frac{\kappa e^{-2\tau}}{2D(1-e^{-2\tau})} + \frac{(\kappa - K)\tau e^{-2\tau}}{D(1-e^{-2\tau})^2}, \\ c_{21}^{(0)}(\tau) &= \frac{\kappa e^{-\tau}}{D(1-e^{-2\tau})} - \frac{(\kappa - K)\tau(1+e^{-2\tau})e^{-\tau}}{D(1-e^{-2\tau})^2}, \\ c_{22}^{(0)}(\tau) &= - \frac{\kappa}{2D(1-e^{-2\tau})} + \frac{(\kappa - K)\tau e^{-2\tau}}{D(1-e^{-2\tau})^2} \end{aligned} \tag{58}$$

and the first-order expansion coefficients (50) are functions of the dimensionless variable $\tau = K(t_b - t_a)$. Note that using Eq. (50) in Eq. (57) necessitates the substitution of κ by K.

We now eliminate the dependence of the cumulant expansion (57) from the artificially introduced trial damping constant K. According to the principle of minimal sensitivity [18], we minimize its influence on $W^{(1)}(x_b t_b | x_a t_a ; K)$ by searching for local extrema of $W^{(1)}(x_b t_b | x_a t_a ; K)$ with respect to K, i.e., from the condition

$$\left. \frac{\partial W^{(1)}(x_b t_b ; x_a t_a ; K)}{\partial K} \right|_{K = K^{(1)}(x_b t_b ; x_a t_a)} = 0. \tag{59}$$

It may happen that this equation is not solvable within a certain region of the parameters x_b, t_b, x_a, t_a. In this case, in accordance with the principle of minimal sensitivity [1,17], we look for turning points instead, i.e., we determine the variational parameter $K^{(1)}(x_b t_b ; x_a t_a)$ by solving

$$\left. \frac{\partial^2 W^{(1)}(x_b t_b ; x_a t_a ; K)}{\partial K^2} \right|_{K = K^{(1)}(x_b t_b ; x_a t_a)} = 0. \tag{60}$$

The solution $K^{(1)}(x_b t_b ; x_a t_a)$ from Eqs. (59) or (60) yields the variational result

$$\begin{aligned} & P(x_b t_b | x_a t_a) \\ & \approx \frac{\exp\{ W[x_b t_b ; x_a t_a ; K^{(1)}(x_b t_b ; x_a t_a)] \}}{\int_{-\infty}^{+\infty} dx_b \exp\{ W[x_b t_b ; x_a t_a ; K^{(1)}(x_b t_b ; x_a t_a)] \}} \end{aligned} \tag{61}$$

for the conditional probability density. Note that variational perturbation theory does not preserve the normalization of the conditional probability density. Although the perturbative result (49),(50) is still normalized, in the usual sense (7), to first order in the coupling constant g, this normalization is spoiled by choosing an x_b-dependent damping constant $K^{(1)}(x_b t_b ; x_a t_a)$. Thus, we have to normalize the variational conditional probability density, according to Eq. (61), at the end (compare the similar situation for the variational ground-state wave function in Refs. [17],[19]).

We have applied variational perturbation theory to analyze the nonlinear stochastic model (29) with $D=1$, $x_a = t_a = 0$ below and above the laser threshold, i.e., $\kappa = 1$ and $\kappa = -1$, for small and strong coupling $g = 0.1$ and $g = 10$. The results for the conditional probability density $P(x_b t_a | x_a t_a)$ are plotted in Figs. 2 and 3. On the scale of the figures, they show no significant deviation from numerical solutions of the corresponding Fokker-Planck equation. Some minor deviations only occur above the laser threshold $\kappa = -1$ for g

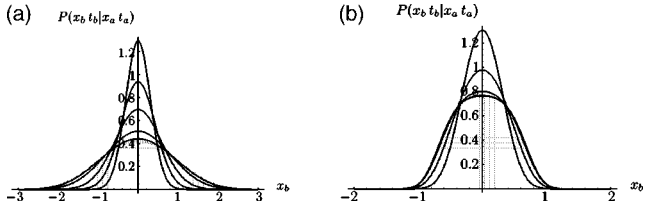


FIG. 2. Conditional probability density $P(x_b t_b | x_a t_a)$ below the laser threshold at $\kappa = 1$, $D = 1$, and $x_a = t_a = 0$. For the coupling constants $g = 0.1$ in (a) and $g = 10$ in (b) the distribution is shown for times $t_b = 0.05$, 0.1, 0.2, 0.5, 1 from top to bottom at the origin, respectively.

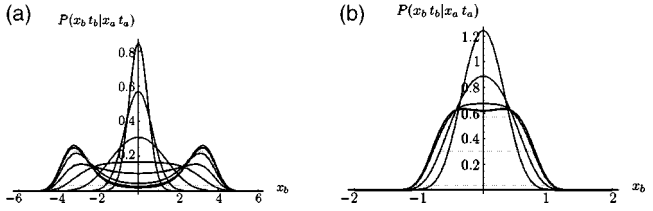


FIG. 3. Conditional probability density $P(x_b t_b | x_a t_a)$ above the laser threshold at $\kappa = -1$, $D = 1$, and $x_a = t_a = 0$. For the coupling constants $g = 0.1$ in (a) and $g = 10$ in (b) the distribution is shown for times $t_b = 0.1$, 0.2, 0.5, 1, 1.5, 2, 3, 4 and $t_b = 0.05$, 0.1, 0.2, 0.5 from top to bottom at the origin, respectively.

$=0.1$ [20]. Both figures illustrate how the densities, which originally peaked at the origin, turn into their stationary solutions in the long-time limit. In Fig. 2, the stationary solution reveals one extremum below the threshold, whereas Fig. 3 reveals two extrema above the laser threshold.

V. CONCLUSION

We have presented the lowest-order variational calculation for the conditional probability density of the nonlinear drift coefficient (29). By going to higher orders in variational perturbation theory, it is straightforward to systematically increase the accuracy to any desired degree [1].

Note Added. Since completing this work, we have become aware of the alternative approach [21] to variationally analyzing the nonlinear model (29). Above the laser threshold $\kappa=-1$, that method yields for all times t_b a unique solution of the extremal condition corresponding to Eq. (59). However, the resulting conditional probability density shows for larger times t_b significant deviations from our, and from numerical solutions of the Fokker-Planck equation.

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