2.4 BRA-KET (DIRAC) FORMALISM

2.4.1 VECTORS IN HILBERT SPACE

Any quantum dynamical state of a physical system may be represented by a vector (*bra* or *ket*) with a unitary norm (see below) within the so-called *space of the quantum states* or the *Hilbert space*:

$$\begin{array}{ccc} & \textit{QUANTUM} & \\ & \textit{DYNAMICAL} & \\ & \textit{STATE} & \\ \swarrow & & \searrow \\ \langle\psi| \in \bar{\mathcal{H}} & \overset{\textit{DUAL}}{\underset{\textit{SPACES}}{\leftrightarrow}} & |\psi\rangle \in \mathcal{H} \\ \textit{bra}-\textit{vectors} & & \textit{ket}-\textit{vectors} \end{array}$$

The Hilbert space is a vectorial space with *scalar product*, which is *complete*. A metrical space is called *complete* (or Banach space) if any convergent sequence of space elements has its limits within the space. On the other side, the scalar product is defined as the functional constructed on an abelian (commutative) vectorial space $\mathcal{V}$

$$\langle\ |\ \rangle : \mathcal{V}\times\mathcal{V} \to \mathbf{C} \tag{2.278}$$

with properties

I. *The norm; the positive definite self scalar product*:

$$\left\|\langle\psi|\psi\rangle\right\|^2 = \langle\psi|\psi\rangle \geq 0\ ;\ \langle\psi|\psi\rangle = 0 \Leftrightarrow |\psi\rangle = 0 \tag{2.279}$$

II. *Ket-Distributivity*:

$$\langle\psi|a_1\alpha_1 + a_2\alpha_2\rangle = a_1\langle\psi|\alpha_1\rangle + a_2\langle\psi|\alpha_2\rangle \tag{2.280}$$

III. *Hermiticity*:

$$\langle\psi|\alpha\rangle = \overline{\langle\alpha|\psi\rangle} \tag{2.281}$$

These properties allow the following consequences:

Ibis. *The null vector property*:

$$|0\rangle = 0|\psi\rangle, \forall|\psi\rangle \in \mathcal{H} \tag{2.282}$$

IIbis. *Superposition property*:

$$\langle\psi|a_1\alpha_1 + a_2\alpha_2\rangle = \langle\psi|\left(a_1|\alpha_1\rangle + a_2|\alpha_2\rangle\right), \forall|\psi\rangle,|\alpha_1\rangle,|\alpha_2\rangle \in \mathcal{H} \tag{2.283}$$

IIIbis. *Bra-Distributivity*:

$$\begin{aligned}\langle a_1\alpha_1 + a_2\alpha_2|\psi\rangle &= \overline{\langle\psi|a_1\alpha_1 + a_2\alpha_2\rangle} = \overline{a_1\langle\psi|\alpha_1\rangle + a_2\langle\psi|\alpha_2\rangle}\\ &= a_1^*\overline{\langle\psi|\alpha_1\rangle} + a_2^*\overline{\langle\psi|\alpha_2\rangle} = a_1^*\langle\alpha_1|\psi\rangle + a_2^*\langle\alpha_2|\psi\rangle\end{aligned} \tag{2.284}$$

I2bis. *Schwartz inequality*:

$$\left\|\langle\alpha|\beta\rangle\right\|^2 \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle, \ \forall|\alpha\rangle,|\beta\rangle \in \mathcal{H} \tag{2.285}$$

Proof: using the above rules one can consider the successive equivalences:

$$\begin{aligned}&\langle\alpha + a\beta|\alpha + a\beta\rangle \ge 0, \ \forall a \in \mathbf{R}\\ &\Leftrightarrow \left(\langle\alpha| + a\langle\beta|\right)\left(|\alpha\rangle + a|\beta\rangle\right) \ge 0\\ &\Leftrightarrow \langle\alpha|\alpha\rangle + a[\langle\alpha|\beta\rangle + \underbrace{\langle\beta|\alpha\rangle}_{\overline{\langle\alpha|\beta\rangle}}] + a^2\langle\beta|\beta\rangle \ge 0\\ &\Leftrightarrow a^2\langle\beta|\beta\rangle + 2a\,\mathrm{Re}\left(\langle\alpha|\beta\rangle\right) + \langle\alpha|\alpha\rangle \ge 0\\ &\Leftrightarrow \Delta \le 0 : \mathrm{Re}^2\left(\langle\alpha|\beta\rangle\right) - \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \le 0\\ &\Leftrightarrow \mathrm{Re}^2\left(\langle\alpha|\beta\rangle\right) \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle, \ \forall|\beta\rangle \in \mathcal{H}\end{aligned} \tag{2.286}$$

therefore true also for the specialization:

$$|\beta'\rangle = e^{i\theta}|\beta\rangle ; \ \langle\beta'| = \langle\beta|e^{-i\theta} \tag{2.287}$$

providing the further inequality:

$$\operatorname{Re}\left(e^{i\theta}\langle\alpha|\beta\rangle\right) \le \sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle} \tag{2.288}$$

Employing the left side expression only, one gets:

$$\begin{aligned}
\operatorname{Re}\left(e^{i\theta}\langle\alpha|\beta\rangle\right) &= \operatorname{Re}\left((\cos\theta + i\sin\theta)\langle\alpha|\beta\rangle\right)\\
&= \operatorname{Re}\left[(\cos\theta + i\sin\theta)\left(\operatorname{Re}\langle\alpha|\beta\rangle + i\operatorname{Im}\langle\alpha|\beta\rangle\right)\right]\\
&= \operatorname{Re}\begin{bmatrix}\cos\theta\operatorname{Re}\langle\alpha|\beta\rangle - \sin\theta\operatorname{Im}\langle\alpha|\beta\rangle\\ +i\left(\cos\theta\operatorname{Im}\langle\alpha|\beta\rangle + \sin\theta\operatorname{Re}\langle\alpha|\beta\rangle\right)\end{bmatrix}\\
&= \cos\theta\operatorname{Re}\langle\alpha|\beta\rangle - \sin\theta\operatorname{Im}\langle\alpha|\beta\rangle \ , \forall\theta
\end{aligned} \tag{2.289}$$

While choosing θ so that

$$\tan\theta = -\frac{\operatorname{Im}\langle\alpha|\beta\rangle}{\operatorname{Re}\langle\alpha|\beta\rangle} \tag{2.290}$$

we firstly have:

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \sqrt{\frac{\operatorname{Re}^2\langle\alpha|\beta\rangle}{\left\|\langle\alpha|\beta\rangle\right\|^2}} = \frac{\operatorname{Re}\langle\alpha|\beta\rangle}{\left\|\langle\alpha|\beta\rangle\right\|} \tag{2.291}$$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = -\frac{\operatorname{Im}\langle\alpha|\beta\rangle}{\left\|\langle\alpha|\beta\rangle\right\|} \tag{2.292}$$

and then

$$\operatorname{Re}\left(e^{i\theta}\langle\alpha|\beta\rangle\right) = \frac{\operatorname{Re}^2\langle\alpha|\beta\rangle + \operatorname{Im}^2\langle\alpha|\beta\rangle}{\left\|\langle\alpha|\beta\rangle\right\|} = \left\|\langle\alpha|\beta\rangle\right\| \tag{2.293}$$

that replaced in the last inequality proofs the Schwartz theorem.

I3bis. *The triangle inequality theorem*:

$$\left\||\alpha\rangle + |\beta\rangle\right\| \le \left\||\alpha\rangle\right\| + \left\||\beta\rangle\right\| \tag{2.294}$$

may be immediately proofed with the help of Schwartz's one equivalent forms by the chain of relations:

$$\left\| |\alpha\rangle + |\beta\rangle \right\|^2 \le \left(\left\| |\alpha\rangle \right\| + \left\| |\beta\rangle \right\| \right)^2$$

$$\Leftrightarrow \left(\langle\alpha| + \langle\beta| \right)\left(|\alpha\rangle + |\beta\rangle \right) \le \langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle + 2\sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}$$

$$\mathrm{Re}\left(\langle\alpha|\beta\rangle \right) \le \sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle} \tag{2.295}$$

that recovers one of the above Schwartz proof's inequality.

I4bis. *Cosines definition between states' vectors*

$$\cos\left(|\alpha\rangle, |\beta\rangle \right) \equiv \frac{\langle\alpha|\beta\rangle}{\sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta|\beta\rangle}} \le 1, \ \forall |\alpha\rangle, |\beta\rangle \in \mathcal{H} \tag{2.296}$$

appears as a natural consequence of the Schwartz theorem. However, the validity of this definition may be seen also by considering the metric distance between two vectors in Hilbert space releasing with the generalized cosines theorem:

$$\begin{aligned} d\left(|\alpha\rangle, |\beta\rangle \right) &= \left\| |\alpha\rangle - |\beta\rangle \right\| = \left\| |\alpha - \beta\rangle \right\| = \sqrt{\langle\alpha - \beta|\alpha - \beta\rangle} \\ &= \sqrt{\langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle - \left(\langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle \right)} \\ &= \sqrt{\langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle - 2\,\mathrm{Re}\langle\alpha|\beta\rangle} \end{aligned} \tag{2.297}$$

where one can recognize the classical scalar product definition generalized through the present Hilbert states' vectors:

$$\langle\alpha|\beta\rangle = \left\| |\alpha\rangle \right\| \left\| |\beta\rangle \right\| \cos\left(|\alpha\rangle, |\beta\rangle \right) = \sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta|\beta\rangle} \frac{\langle\alpha|\beta\rangle}{\sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta|\beta\rangle}} \tag{2.298}$$

In next, the vector states are to be combined with operators to provide further transformations on Hilbert space of quantum reality.

2.4.2 LINEAR OPERATORS IN HILBERT SPACE

The basic definition of the linear operators acting on Hilbert space reads as:

$$\hat{A}: \mathcal{H} \to \mathcal{H}_1 \subset \mathcal{H} \tag{2.299}$$

They have the *linearity* role in quantum (eigen) equations,

$$\hat{A}\left(a_1|\alpha_1\rangle + a_2|\alpha_2\rangle\right) = a_1\hat{A}|\alpha_1\rangle + a_2\hat{A}|\alpha_2\rangle \tag{2.300}$$

and in *observability* through transforming the observables' averages or the transition probabilities by the conjugate property:

$$\overline{\langle\alpha|\hat{A}|\beta\rangle} = \langle\beta|\hat{A}^+|\alpha\rangle \tag{2.301}$$

Such feature implies important consequences in bra-ket formalism. For instance, if one has the operatorial-ket equation

$$\hat{A}|\alpha\rangle = |\psi\rangle \tag{2.302}$$

the corresponding bra-equation is found through equivalences

$$\overline{\langle\beta|\hat{A}|\alpha\rangle} = \overline{\langle\beta|\psi\rangle} \Leftrightarrow \langle\alpha|\hat{A}^+|\beta\rangle = \langle\psi|\beta\rangle$$
$$\Rightarrow \langle\psi| = \langle\alpha|\hat{A}^+ \tag{2.303}$$

In the same manner further operatorial properties may be unfolded, as follows.

I. *double conjugation*

$$\left(\hat{A}^+\right)^+ = \hat{A} \tag{2.304}$$

$$\langle\alpha|\left(\hat{A}^+\right)^+|\beta\rangle = \overline{\langle\beta|\hat{A}^+|\alpha\rangle} = \overline{\overline{\langle\alpha|\hat{A}|\beta\rangle}} = \langle\alpha|\hat{A}|\beta\rangle \tag{2.305}$$

II. *observables' product conjugation*

$$\left(\hat{A}\hat{B}\right)^+ = \hat{B}^+\hat{A}^+ \tag{2.306}$$

$$\langle\alpha|\left(\hat{A}\hat{B}\right)^+|\beta\rangle = \overline{\underbrace{\langle\beta|\hat{A}}_{\langle\phi|}\underbrace{\hat{B}|\alpha\rangle}_{|\psi\rangle}} = \overline{\langle\phi|\psi\rangle} = \langle\psi|\phi\rangle = \underbrace{\langle\alpha|\hat{B}^+}_{\langle\psi|}\underbrace{\hat{A}^+|\beta\rangle}_{|\phi\rangle} \tag{2.307}$$

that can be by induction generalized to the rule:

$$\left(\hat{A}_1 \ldots \hat{A}_n\right)^+ = \hat{A}_n^+ \ldots \hat{A}_1^+ \tag{2.308}$$

III. *distributivity conjugation*

$$\left(a\hat{A} + b\hat{B}\right)^+ = a^* \hat{A}^+ + b^* \hat{B}^+ \tag{2.309}$$

$$\begin{aligned}\langle\alpha|\left(a\hat{A} + b\hat{B}\right)^+|\beta\rangle &= \overline{\langle\beta|\left(a\hat{A} + b\hat{B}\right)|\alpha\rangle} = \overline{a\langle\beta|\hat{A}|\alpha\rangle + b\langle\beta|\hat{B}|\alpha\rangle} \\ &= a^*\overline{\langle\beta|\hat{A}|\alpha\rangle} + b^*\overline{\langle\beta|\hat{B}|\alpha\rangle} = a^*\langle\alpha|\hat{A}^+|\beta\rangle + b^*\langle\alpha|\hat{B}^+|\beta\rangle \\ &= \langle\alpha|a^*\hat{A}^+ + b^*\hat{B}^+|\beta\rangle\end{aligned} \tag{2.310}$$

Next, the hermiticity property of operators is formalized in direct way through fulfilling the auto-adjunct condition:

$$\hat{A}^+ = \hat{A} \tag{2.311}$$

while *anti-hermitic* operators behave like:

$$\hat{A}^+ = -\hat{A} \tag{2.312}$$

The direct consequences regard the hermiticity and anti-hermiticity of the next combinations:

IV. $\hat{A} + \hat{A}^+$ – *hermitic*:

$$\left(\hat{A} + \hat{A}^+\right)^+ = \hat{A}^+ + \left(\hat{A}^+\right)^+ = \hat{A} + \hat{A}^+ \tag{2.313}$$

V. $\hat{A} - \hat{A}^+$ – *anti-hermitic*:

$$\left(\hat{A} + (-1)\hat{A}^+\right)^+ = \hat{A}^+ + (-1)\left(\hat{A}^+\right)^+ = -\left(\hat{A} - \hat{A}^+\right) \tag{2.314}$$

VI. $\pm i\left(\hat{A}-\hat{A}^{+}\right)$ – *hermitic*:

$$\left[\pm i\left(\hat{A}+(-1)\hat{A}^{+}\right)\right]^{+}=\mp i\left[\hat{A}^{+}+(-1)\left(\hat{A}^{+}\right)^{+}\right]=\pm i\left(\hat{A}-\hat{A}^{+}\right) \quad (2.315)$$

VII. Any linear operator $\hat{F}$ may be decomposed on *one hermitic and one anti-hermitic* or on *only hermitic operators* as the context demands:

$$\begin{aligned}\hat{F}&=\frac{1}{2}\left(\hat{F}+\hat{F}^{+}+\hat{F}-\hat{F}^{+}\right)\\&=\underbrace{\frac{1}{2}\left(\hat{F}+\hat{F}^{+}\right)}_{hermitic}+\underbrace{\frac{1}{2}\left(\hat{F}-\hat{F}^{+}\right)}_{anti-hermitic}=\underbrace{\frac{1}{2}\left(\hat{F}+\hat{F}^{+}\right)}_{hermitic}+i\underbrace{\left[-\frac{1}{2}i\left(\hat{F}-\hat{F}^{+}\right)\right]}_{hermitic}\end{aligned} \quad (2.316)$$

With these there can be constructed the so called *linear operators' algebra* (LOA): $\{+_{\text{operators,}}\bullet_{\text{with scalars,}}\bullet_{\text{with operators}}\}$ on the Hilbert space of state vectors respecting the following operations:

LOA-I. *the sum of operators*:

$$\left(\hat{A}+\hat{B}\right)|\alpha\rangle=\hat{A}|\alpha\rangle+\hat{B}|\alpha\rangle \quad (2.317)$$

LOA-II. *the product of operators with scalars*:

$$\left(a\hat{A}\right)|\alpha\rangle=a\left(\hat{A}|\alpha\rangle\right) \quad (2.318)$$

LOA-III. *the product among operators*:

$$\left(\hat{A}\hat{B}\right)|\alpha\rangle=\hat{A}\left(\hat{B}|\alpha\rangle\right) \quad (2.319)$$

However, note that this algebra is non-commutative, the measure of this non-commutativity being the introduced *commutator*

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A} \quad (2.320)$$

and the anti-commutator

$$\left\{\hat{A},\hat{B}\right\}=\hat{A}\hat{B}+\hat{B}\hat{A} \quad (2.321)$$

of two operators, respectively.

If the operators $\hat{A}\,\&\,\hat{B}$ are hermitic then the bellow assertions holds:

$$\left\{\hat{A},\hat{B}\right\}\text{-}hermitic;\ \left[\hat{A},\hat{B}\right]\text{-}anti\text{-}hermitic;\ i\left[\hat{A},\hat{B}\right]\text{-}hermitic \quad (2.322)$$

as may be immediately proofed. Moreover, with the same assumption there is true that the product $\hat{A}\hat{B}$ is hermitic if the operators $\hat{A}\,\&\,\hat{B}$ commute:

$$\underbrace{\left(\hat{A}\hat{B}\right)^{*}}_{\mapsto\cdots}=\hat{B}^{*}\hat{A}^{*}=\hat{B}\hat{A}=\underbrace{\hat{A}\hat{B}}_{\cdots\uparrow}+\underbrace{\left[\hat{B},\hat{A}\right]}_{0} \quad (2.323)$$

while $\hat{A}^{+}\hat{A}\,\&\,\hat{A}\hat{A}^{+}$ are hermitic combination for any operators (non-necessary hermitic) $\hat{A}$. As well, there is interesting to note that if an operator $\hat{C}$ is hermitic, then any combination $\hat{G}=\hat{A}\hat{C}\hat{A}^{+}$, for any arbitrary operator $\hat{A}$, is as well hermitic:

$$\hat{G}^{+}=\left(\hat{A}\hat{C}\hat{A}^{+}\right)^{+}=\hat{A}\hat{C}^{+}\hat{A}^{+}=\hat{A}\hat{C}\hat{A}^{+}=\hat{G} \quad (2.324)$$

Back to commutators and anti-commutators, there is useful quoting their main properties

$$\left[\hat{A},\hat{B}\right]=-\left[\hat{B},\hat{A}\right] \quad (2.325)$$

$$\left[\hat{A},\left[\hat{B},\hat{C}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]=0\ \text{(the Jacobi identity)} \quad (2.326)$$

$$\left[\hat{A},\hat{B}+\hat{C}\right]=\left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right] \quad (2.327)$$

$$\left[\hat{A},a\hat{B}\right]=\left[a\hat{A},\hat{B}\right]=a\left[\hat{A},\hat{B}\right] \quad (2.328)$$

$$\left[\hat{A},\hat{B}_1\hat{B}_2\right]=\left[\hat{A},\hat{B}_1\right]\hat{B}_2+\hat{B}_1\left[\hat{A},\hat{B}_2\right] \quad (2.329)$$

$$\left[\hat{A},\hat{B}_1\hat{B}_2...\hat{B}_n\right]=\left[\hat{A},\hat{B}_1\right]\hat{B}_2...\hat{B}_n+\hat{B}_1\left[\hat{A},\hat{B}_2\right]\hat{B}_3...\hat{B}_n+\hat{B}_1...\hat{B}_{n-1}\left[\hat{A},\hat{B}_n\right] \quad (2.330)$$

$$\left[\hat{A},\hat{B}_1\hat{B}_2...\hat{B}_n\right]=\left\{\hat{A},\hat{B}_1\right\}\hat{B}_2...\hat{B}_n-\hat{B}_1\left\{\hat{A},\hat{B}_2\right\}\hat{B}_3...\hat{B}_n$$
$$+\left(-1\right)^{n-1}\hat{B}_1...\hat{B}_{n-1}\left\{\hat{A},\hat{B}_n\right\} \tag{2.331}$$

as one may show directly or by induction based on the above definitions and properties.

Finally, we are introducing the so-called *functions of operators*, based on the analyticity of the complex functions that are expanded in the Taylor series:

$$f:\mathbf{C}\to\mathbf{C}\Big|\ f(z)=f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}z^n \tag{2.332}$$

which allows in the base of the formal correspondence:

$$z^n\to\hat{A}^n \tag{2.333}$$

the operatorial function expansion:

$$f(\hat{A})=f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}\hat{A}^n \tag{2.334}$$

Some examples for operatorial function expansion are:

$$\frac{1}{\hat{1}-\hat{A}}=1+\hat{A}+\hat{A}^2+...+\hat{A}^n+... \tag{2.335}$$

$$\exp\left(\hat{A}\right)=\sum_{n=0}^{\infty}\frac{\hat{A}^n}{n!}=1+\hat{A}+\frac{\hat{A}^2}{2!}+...+\frac{\hat{A}^n}{n!}+... \tag{2.336}$$

The last expansion helps in proving the important identity:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}}=\hat{B}+\sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{\left[\hat{A},\left[\hat{A},...\left[\hat{A},\hat{B}\right]\right]\right]}_{n\ \ parentheses}=\hat{B}+\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]+... \tag{2.337}$$

Restricting to the second order only, one can arrange successively that

$$
\begin{aligned}
e^{\hat{A}}\hat{B}e^{-\hat{A}} &= \left(\hat{1}+\hat{A}+\frac{\hat{A}^2}{2!}+\frac{\hat{A}^3}{3!}+...\right)\hat{B}\left(\hat{1}-\hat{A}+\frac{\hat{A}^2}{2!}-\frac{\hat{A}^3}{3!}+...\right)\\
&= \hat{B}+\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)+\left(\frac{\hat{A}^2}{2!}\hat{B}+\hat{B}\frac{\hat{A}^2}{2!}-\hat{A}\hat{B}\hat{A}\right)+...\\
&= \hat{B}+\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left(\hat{A}\hat{A}\hat{B}+\hat{B}\hat{A}\hat{A}-2\hat{A}\hat{B}\hat{A}\right)+...\\
&= \hat{B}+\left[\hat{A},\hat{A}\right]+\frac{1}{2}\left(\hat{A}\left[\hat{A},\hat{B}\right]+\left[\hat{B},\hat{A}\right]\hat{A}\right)+...\\
&= \hat{B}+\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left(\hat{A}\left[\hat{A},\hat{B}\right]-\left[\hat{A},\hat{B}\right]\hat{A}\right)+...\\
&= \hat{B}+\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]+... \qquad (2.338)
\end{aligned}
$$

while for the higher terms the induction method can be applied.

Such expansion is extremely useful in showing that for operators, in general we have

$$\exp\left(\hat{A}\right)\exp\left(\hat{B}\right)\neq\exp\left(\hat{A}+\hat{B}\right) \qquad (2.339)$$

In fact, the left hand side product can be evaluated through considering the more general expression, say:

$$f(\lambda)=\exp\left(\lambda\hat{A}\right)\exp\left(\lambda\hat{B}\right),\ \lambda\in\mathbf{R} \qquad (2.340)$$

that through derivation provides expression:

$$\frac{df(\lambda)}{d\lambda}=\hat{A}\exp\left(\lambda\hat{A}\right)\exp\left(\lambda\hat{B}\right)+\underbrace{\exp\left(\lambda\hat{A}\right)\hat{B}}_{?}\exp\left(\lambda\hat{B}\right) \qquad (2.341)$$

to be then integrated. However, one recognizes that through reconsidering the above formula within the actual parameter involvement, namely

$$
\begin{aligned}
f(\lambda)=e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} &= \hat{B}+\lambda\left[\hat{A},\hat{B}\right]+\frac{\lambda^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\\
&+...\frac{\lambda^n}{n!}\underbrace{\left[\hat{A},\left[\hat{A},...\left[\hat{A},\hat{B}\right]\right]\right]}_{n\ \ parenthesis}+... \qquad (2.342)
\end{aligned}
$$

is obtained that

$$\exp\left(\lambda\hat{A}\right)\hat{B}=\left\{\begin{array}{l}\hat{B}+\lambda\left[\hat{A},\hat{B}\right]+\frac{\lambda^{2}}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\\+...\frac{\lambda^{n}}{n!}\left[\hat{A},\left[\hat{A},...\left[\hat{A},\hat{B}\right]\right]\right]+...\end{array}\right\}\exp\left(\lambda\hat{A}\right)\tag{2.343}$$

helping in rewriting the above derivative as:

$$\frac{df(\lambda)}{d\lambda}=\left\{\begin{array}{l}\left(\hat{A}+\hat{B}\right)+\lambda\left[\hat{A},\hat{B}\right]+\frac{\lambda^{2}}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\\+...\frac{\lambda^{n}}{n!}\left[\hat{A},\left[\hat{A},...\left[\hat{A},\hat{B}\right]\right]\right]+...\end{array}\right\}e^{\lambda\hat{A}}e^{\lambda\hat{B}}\tag{2.344}$$

Integration of this expression, respecting the parameter λ, while keeping in mind that $f(0)=1$, provides the searched result:

$$\begin{aligned}f(\lambda)&=\exp\left(\lambda\hat{A}\right)\exp\left(\lambda\hat{B}\right)\\&=\exp\left\{\begin{array}{l}\left(\hat{A}+\hat{B}\right)\lambda+\frac{\lambda^{2}}{2!}\left[\hat{A},\hat{B}\right]+\frac{\lambda^{3}}{3!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\\+...\frac{\lambda^{n+1}}{(n+1)!}\left[\hat{A},\left[\hat{A},...\left[\hat{A},\hat{B}\right]\right]\right]+...\end{array}\right\}\end{aligned}\tag{2.345}$$

producing the particularization for $\lambda=1$:

$$\exp\left(\hat{A}\right)\exp\left(\hat{B}\right)=\exp\left\{\begin{array}{l}\left(\hat{A}+\hat{B}\right)+\frac{1}{2}\left[\hat{A},\hat{B}\right]+\frac{1}{6}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\\+...\frac{1}{(n+1)!}\underbrace{\left[\hat{A},\left[\hat{A},...\left[\hat{A},\hat{B}\right]\right]\right]}_{n\ \ parenthesis}+...\end{array}\right\}\tag{2.346}$$

or, for the particular case in which $\hat{A}$ commutes with $\left[\hat{A},\hat{B}\right]$ the further simplification is obtained as (the so called Baker-Hausdorff formula):

$$\exp\left(\hat{A}\right)\exp\left(\hat{B}\right)=\exp\left\{\left(\hat{A}+\hat{B}\right)+\frac{1}{2}\left[\hat{A},\hat{B}\right]\right\}\tag{2.347}$$

that still do not allow the direct summation of operators under exponential function unless they commutes as well.

Both vectors and operators on Hilbert space of quantum states admit also various representations with which occasion additional properties should be revealed, as will be in next section exposed.

2.4.3 SPECTRAL REPRESENTATIONS OF VECTORS AND OPERATORS

Let be a vectorial (linear) finite space with scalar (dot) product operation included, $\mathcal{V}/\mathrm{C}$, with the (ortho-normal) basis or vectors:

$$\left\{\left|e_i\right\rangle\right\}_{1,\ldots,n=\dim\mathcal{V}} \left| \left\langle e_i \middle| e_j \right\rangle = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \right. \tag{2.348}$$

with the column matrix representation:

$$\left|e_1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \left|e_2\right\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots,\ \left|e_n\right\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{2.349}$$

such that any other vector $\left|\alpha\right\rangle$ of the space will be represented on this basis by the linear decomposition:

$$\left|\alpha\right\rangle = \alpha_1\left|e_1\right\rangle + \alpha_2\left|e_2\right\rangle + \ldots + \alpha_n\left|e_n\right\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \sum_{k=1}^{n} \alpha_k \left|e_k\right\rangle \tag{2.350}$$

while its conjugation looks like

$$\begin{aligned} \left\langle\alpha\right| = \overline{\left|\alpha\right\rangle} &= \overline{\alpha_1\left|e_1\right\rangle + \alpha_2\left|e_2\right\rangle + \ldots + \alpha_n\left|e_n\right\rangle} \\ &= \alpha_1^*\left\langle e_1\right| + \alpha_2^*\left\langle e_2\right| + \ldots + \alpha_n^*\left\langle e_n\right| = \begin{pmatrix} \alpha_1^* & \alpha_2^* & \cdots & \alpha_n^* \end{pmatrix} \\ &= \sum_{k=1}^{n} \alpha_k^* \left\langle e_k\right| \end{aligned} \tag{2.351}$$